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MATHEMATICAL FORMALIZATION OF INSTITUTIONS

A. YU. OBYDENOV

PhD in Economics,
Associate Professor,
Department of Management,
Financial University
under the Government of
the Russian Federation.
Research interests:
strategic management, new
institutional economics,
systems approach to
management, complexity
theory in strategic
management.

Email:
alexander.obydenov@gmail.com

ABSTRACT

The article proposes the way of mathematical formalization of institutions in the framework of game-theory approach. The rules are represented by transpositions of payoffs in payment matrix. Such the transitions call the change the structure of game equilibriums. This approach allows separation of coordination and distribution aspects of institutions, division institutions into classes and distinguishing aggregatec of institutions, which produce identical transformation. Through evolutionary game theory formalism it is shown that such parametric management influences the structure of steady states, what allows investigation of institutions' influence on self-organization, in particular, through transformation of topology of phase portrait of the system. In accordance with the fact that dynamic with multiple steady states is typical for coordination problem as a whole, an approach suggested of the rules formalization appears to have rather broad application.

KEY WORDS   INSTITUTIONS, RULES, GAME THEORY, PAYOFF TRANSPOSITION, MATRIX, PARAMETRIC MANAGEMENT

INTRODUCTION

In previous studies we proposed an approach for strategic management based on parametric management where the aspiration to achieve targeted sustainable states and functioning modes is provided by formal institutions [Obydenov A.Yu., 2016b; 2017].

The approach, which considers institutions as a tool of parametric control, sets the task for finding appropriate ways of institutions' mathematical formalization[1] through their mathematical representation. There appear the problem of formal mapping of a set of institutions onto a set of their representations. Our proposed method of institutions' mathematical mapping can facilitate the institutions' mathematical formalization for the purposes of parametric strategic management.

All ways of describing institutions are divided into explicit and implicit. In the first case, formal components of institutions are described, in the second one, quantitative and qualitative economic consequences of the institutions' use. The first examples of an explicit

description of institutions are contained in the Herodotus' treatises (see, example: [Herodotus, 1972]). One of the modern attempts of explicit algebraic formalization of the institution was undertaken by V. Tambovtsev (2004).

An implicit form of describing an institution is specific for microeconomic analysis. The example of such description is analyzed, for instance, in the study of D. Bromley [Bromley D. W., 1989, p. 111–115]. We used the analytical microeconomic approach to determine the consequences of institution's self-regulation [Kryuchkova P.V., Obydenov A.Yu., 2003]. The game-theoretic way of describing institutions is also possible: institutions are taken as fixed rules [Shubik M., 1982], represented by stable strategies in repeating games [Schotter A., 1981]. However, such approaches do not allow to consider institutions as parameters that control the object state (the ratio of gains in the payment matrix). The proposed method of both explicit and implicit algebraic representation of institutions is relevant due to the fact that it meets the requirements of the parametric approach [A. Obydenov, 2016b; 2017].

ALGEBRAIC FORMALIZATION OF INSTITUTIONS. DISCRETE MODEL

We begin with an implicit approach. It proposes the representation of rules through the consequences of their application in matrix games. Consider the case of a symmetric game 2x2:

x_1, x_1	x_3, x_2
x_2, x_3	x_4, x_4

We consider that the applying of the rule leads to changes in the distribution of gains in the payment matrix. You can represent x_i, $i = \overline{1,4}$ as independent coordinates of an algebraic vector in the phase space:

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}. \quad (1)$$

X will define the state of the system, but the action of the rule is the possible transition of the system from one state to another. We represent the rule as an operator $\hat{R}$, and thus we proceed to an explicit description of the rules.

Let suppose that in a number of cases within the framework of the dynamic problem, in particular, for analyzing equilibrium points in pure strategies, the ratio of elements X is important. That's why it is enough to represent $\hat{R}$ as a permutation of the components of an algebraic vector X.

For $x_1 > x_2 > x_4 > x_3$ we obtain a pure coordination game with two Nash equilibriums (coordination problem[2]) ($X_1 = X$ for $x_1 > x_2 > x_4 > x_3$).

For $x_2 > x_1 > x_4 > x_3$ a "prisoner dilemma" is appeared with a single Nash equilibrium that is non-optimal by Pareto (the problem of cooperation) ($X_2 = X$ for $x_2 > x_1 > x_4 > x_3$).

For $x_1 > x_2 > x_3 > x_4$ we are dealing with a game where the Nash equilibrium coincides with the Pareto equilibrium (the Nash equilibrium is Pareto optimal) ($X_2 = X$ for $x_1 > x_2 > x_3 > x_4$).

Formally, the action of the rule can be represented as:
$$\hat{R}X = X', \quad (2)$$
where $\hat{R}$ – is the permutation operator.

We introduce the designation: $\hat{R}_{13} \equiv \hat{R}: X_1 \longmapsto X_3$, then for the above-mentioned cases we have:
$$\hat{R}_{13} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}. \quad (3)$$
The rule matrix is:
$$[\hat{R}_{13}] = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}. \quad (4)$$
Permutations can be represented as disjoint cycles (in this case, as an elementary transposition), from here: $\hat{R}_{13} = (3, 4)$.

So, the rule $\hat{R}_{13}$ converts a parametrically controlled system from state X_1 to state X_3. The rule representation is a permutation operator. You can use the known properties of permutations that automatically apply to the rules.

The permutation matrixes are orthogonal. They form a group.

The reverse matrix is equal to the transposed one. Thus, the rules form a group of transformations. In particular the opposite rule implies, which cancels this. The representation matrix of the inverse rule is transposed with respect to the original one. The rule applied the same number of times, which is equal to the group order, returns the system to the initial state.

It should be noted that the chosen form of the algebraic representation of the rule does not allow us to establish a one-to-one correspondence between the rule and its representation. Generally speaking, each representation is degenerate: it corresponds to a set (class) of rules.

Let consider a permutation, which is characterized by a single operator $\hat{E}$. This permutation will correspond to a whole class of such rules and under which the action on the payment matrix in the form of permutations of payments will leave unchanged the relations between payments.

We assume that from the point of view of the influence on the equilibrium in the game the action of the institutions leaves the ratio of gains unchanged, thereby do not have the coordination economic consequences of the game situation and therefore are purely distributive. Permutations that are representations of the rules leave the constant amount of each player's gains for all possible strategies. From the economic point of view, the expected payoff (weighted average gain) remains unchanged for each participant. Such weighted average gain is equal to one-fourth of the sum of the gains for any participant. The participants' possibility for choosing each of the possible strategies in the game is 0.5. In this context, the action of such rule can be called coordination.

Definition. The action of the rule is called coordinating as weighted average gain. As a result of its application, the sum of each participant's gains for all possible strategies remains unchanged.

The ways of separation the rules effects on coordination and distribution may be different and may be several of them. In this case, the choice of a particular method of division requires an agreement, a convention. The above-mentioned argument suggests one way of this separation.

CONTINUOUS KINETIC MODEL

Let it be some event that can take place with the cooperative behavior of particular group members.

Then for calculation the expected participants' gains of following cooperative and non-cooperative strategies can be written as respectively following relationships:
$$EU(K) = nx_1 + (1-n)x_3; \quad (5)$$
$$EU(\overline{K}) = nx_2 + (1-n)x_4, \quad (6)$$
where EU – is the expected gain of the participants; K – cooperative strategy; n – is the share of participants of following cooperative strategy; x_i – gains in the original payment matrix.

The average (weighted) gain participant is:
$$EU_{cp} = nEU(K) + (1-n)EU(\overline{K}) = n^2(x_1 - x_2 - x_3 + x_4) + n(x_3 + x_2 - 2x_4) + x_4. \quad (7)$$

Then, starting from the replication equalization in the framework of evolutionary game theory (in fact this equalization is reflected in the first equality of relation (8)) of the kinetic equalization for n as an independent variable can be written

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[1] The mathematical formalization of institutions is understood as the use of formal methods of describing institutions.

[2] More information about the problem of coordination and other problems in the interaction of economic actors in the framework of game theory, see: [Schotter A., 1981, p. 22-24].

[Hofbauer J., Sigmund K., 2003; Smith J. M., 1982; Golman R., Page S. E., 2010; Bierman, H.S., Fernandez L, 1998, p. 384–397]:

$$\frac{dn}{dt} = n(EU(K) - EU_{cp}) = n(((x_2 - x_1) +$$
$$+ (x_3 - x_4))n^2 + (x_1 - x_2 + 2(x_4 - x_3))n + (x_3 - x_4)) \quad (8)$$

As control parameters we introduce: $a = x_1 - x_2$ and $b = x_3 - x_4$.

Then in the framework of the simultaneous game we have considered above in pure strategies:

• for $a > 0$, $b < 0$ there are two Nash equilibriums (pure coordination game, coordination problem);
• for $a < 0$, $b < 0$ one Nash equilibrium, different from Pareto equilibrium (prisoner's dilemma, problem of cooperation);
• for $a > 0$, $b > 0$ oone Nash equilibrium, coinciding with the Pareto optimal.

And in the framework of the evolutionary game from (8) we get:

$$\frac{dn}{dt} = n((b-a)n^2 + (a-2b)n + b). \quad (9)$$

As the initial situation before applying the parametric control we consider the case $a > 0$, $b < 0$. Then this equalization which is solved relatively has three roots (0, b/(b – a), 1).

If now the equalization

$$\frac{dQ}{dt} = -Q^3 + 3Q^2 - 2Q, \quad (10)$$

where Q – is the volume of released production by the economic entity, which describes the management model of the economic entity behavior in the framework of the strong form of the rationality bounded hypothesis (Obydenov A. Yu., 2017) and normalized to the roots $Q_2 = 0,5$ and $Q_3 = 1$ (instead of roots $Q_2 = 1$ and $Q_3 = 2$), Then instead of this equalization (10) we get:
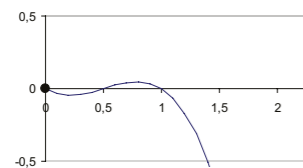
$$\frac{dQ}{dt} = -Q^3 + 1,5Q^2 - 0,5Q. \quad (11)$$

And if in equalization (9) we put $a = 0,5$; $b = -0,5$, then we obtain an equalization up to designation that is identical to equalization (11):

$$\frac{dn}{dt} = -n^3 + 1,5n^2 - 0,5n \quad (12)$$

In the phase diagram we obtain two stable points: $n_1 = 0$, $n_3 = 1$ and one unstable: $n_2 = 0,5$.

Pic 1. Phase diagram corresponding to the relation (12)

Thus, in different ways we obtained identical equalizations with identical roots. In both cases, there are two stable states that create opportunities for parametric control. We know that this case ($a > 0$, $b < 0$) with continuous Q and n corresponds to a simultaneous coordination game in pure strategies with two Nash equilibriums. This example illustrates the "popular theorem": in pure strategies of simultaneous play with strict Nash equilibriums in the evolutionary game within the dynamic's replication correspond to attractors (asymptotically stable states) [Cressman R., 2003].

We put for implementation parametric control in ratio to (12) in (9) $a = 1$, $b = 0$ (one Nash equilibrium, Pareto optimal) and we obtain the kinetic equalization:

$$\frac{dn}{dt} = -n^3 + n^2 \quad (13)$$

We have one stable state $n = 1$:

Pic. 2. Phase diagram corresponding to the relation (13)
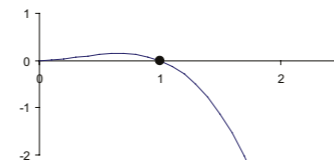
In the framework of the strong form's model hypothesis about the bounded rationality of such control corresponds to price regulation [Obydenov A.Yu., 2017]. From the point of view of game theory with $a = 1$, $b = 0$ we have a simultaneous game with one Nash equilibrium in pure strategies that can be Pareto optimal. Such parametric control corresponds to a rule (institution) with a matrix which specified in equalization (4).

Parametric control within the game

Characteristics and models for comparison	Initial state	State after parametric control
Parameters	$a > 0$, $b < 0$	$a > 0$, $b = 0$
Discrete (non-repeating) game	Pure coordination game, coordination problem, two Nash equilibrium	One Nash equilibrium that can be Pareto optimal
Evolutionary game	Two steady states	One steady state that may be desired
Measures of parametric control in the framework of the model of a strong form of bounded rationality	—	Price regulation

Based on the identity of equalizations (11) and (12), the similarity of the equalization after control (13) we conclude that different tasks in formulation terms lead to identical descriptions, similar control methods and their results. Earlier, we have already noted that similarities between the kinetic equalization in the framework of the strong form model of bounded rationality (equalization 11) and the equation obtained on the basis of evolutionary game theory (equalization 12). Similarities are caused by availability of scale advantage and proportionality to the benefits of changing the independent variable in both cases. In the strong form model of bounded rationality the role of the game-theoretic average gain plays a normal profit per unit of production. In both cases, there is a multiplicity of stable states. All these similarities (see table) allow us to summarize the results of the research.

In general, we can talk about the effect of coordination [Polterovich V.M., 1999, p. 8–9]. For example, if more people follow the mental model, the norm then the less profitable the deviation from the norm becomes and the benefit becomes greater for everyone who adheres to this model of behavior or norm. Some researchers call this effect the "herd effect" [Dixit A. K., Nalebuff B. J., 1991].

Thus, the considered nonlinear dynamics with three stationary states is generally characteristic of the coordination problem and various coordination tasks correspond to such dynamics: for example, the evolution of the traffic direction rule

[Kuzminov Ya. I., Bendukidze KA, Yudkevich MM ., 2006], a model of the evolution of the QWERTY rule, the dynamics of the neighborhood of the white and African American population in American cities [Dixit, Nalebuff, 1991]. Such behavior is specified for active (self-organizing) bistable environments [A. Loskutov, A. S. Mikhailov, 2007]. In this case, our approach may be effective for solving problems of coordination in general.

CONCLUSION

Institutions can be represented as permutations of gains of economic entities within the framework of the game-theoretic approach. Such an algebraic formalization of institutions helps to investigate the properties of institutions through their controlling influence on the economic situation. In particular it will be possible to distinguish more strictly between the coordination and distribution aspects of the institutions' functioning, to identify situations in the practice of managing when a set of established rules leads to an identical transformation and also to combine the rules that produce the same action into classes.

The use of the formalism of the evolutionary theory of games allows us to move from simultaneous discrete games to parametric control in the framework of nonlinear dynamic models. Nonlinear dynamics with three stationary states is found in a wide variety of models. The proposed method of formalizing institutions can be used to solve a sufficiently wide range of tasks, for example, to present institutions as parameters that allow you to manage the self-organization of economic entities strategically in particular by changing the structure of steady states in situations with a problem of coordination.

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